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LETTER TO THE EDITOR

On a mechanical matching condition related to the detection limit for gravitational radiation

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Abstract. Detector characteristics are reduced to four parameters: T_1 = period M_{sys} = total mass, μ = relative transducer mass, r = mismatch ratio. Calculation of the response to a standardized gravity pulse is outlined. Mechanical and electrical signal-to-noise ratios are defined as theoretical limits for zero amplifier noise ($p_a = 0$) and zero detector temperature ($\theta = 0$), respectively. Maximum overall sensibility requires resolving time adjustments depending on r and μ, and proportional to $T_1(p_a/\theta)^{1/2}$. For any suitable μ value, matched systems ($r = 0$) approach the maximum sensibility which is shown to vary according to $T_1 M_{sys}^{1/2}(p_a\theta)^{-1/4}$, contrary to current views based on simplified arguments.

1. Introduction

Current discussion of Weber-type antennae (Weber 1970, Aplin 1972) concentrates on a few particular configurations which are usually treated in terms of a simplified harmonic oscillator model (Gibbons and Hawking 1971, Douglass and Tyson 1971, Drever 1971, *Proc. 6th Int. Conf. on Gravitation and Relativity, Copenhagen*, unpublished). Recently, Rasband *et al* (1972) attempted a quantitative comparison of relative sensitivities between the two extreme geometries of a homogeneous cylinder and a dumbbell. As they did not evaluate the mechanical Q factors nor the electrical signal-to-noise limitations, the question of the absolute sensitivity of differently designed antennae remains unsettled. Furthermore, there is a need for a comprehensive theory covering all intermediate geometries, in view of the rapidly expanding use of divided-bar systems (see also Maeder 1971a, Bramanti and Maischberger 1972, to be published).

The present calculations apply to systems having a central piëzoelectric transducer with an arbitrary cross section ratio,

$$\frac{S_p}{S} = \frac{1-r}{1+r}\left(\frac{Y_x}{Y_p\rho_p}\right)^{1/2} = \frac{1-r}{1+r}\frac{v_x}{v_p\rho_p}. \qquad (1)$$

A mismatch is specified in terms of the reflection coefficient r; Y, ρ, v denote Young's moduli, densities, and sound velocities (quantities with and without the subscript p refer to the transducer and the metal parts, respectively).

2. Basic independent parameters

These are: T_1 = fundamental-mode period, M_{sys} = total system mass ($\equiv M+M_p$),

μ = relative transducer mass ($\equiv M_p/M$), r = force amplitude reflection coefficient (from M_p into M_p). These four parameters determine all detector dimensions, for example the transducer length and the total metal parts length:

$$L_p = v_p T_1 \phi / 2\pi, \qquad L = v T_1 \psi / 2\pi \tag{2}$$

where ϕ and ψ, the respective phaseshifts, are solutions of

$$\tan \frac{\phi\{1+\mu-r(1-\mu)\}}{2\mu(1+r)} = \frac{1-r\cos\phi}{r\sin\phi} \tag{3}$$

$$\psi = \phi(1-r)/\mu(1+r). \tag{4}$$

The respective cross sections are calculated from $S = M_{sys}/\rho L(1+\mu)$ and from equation (1). Keeping T_1, M_{sys}, and μ constant, the geometrical aspects vary as a function of r as depicted in figure 1 for a typical piëzoxyde†–aluminium combination.

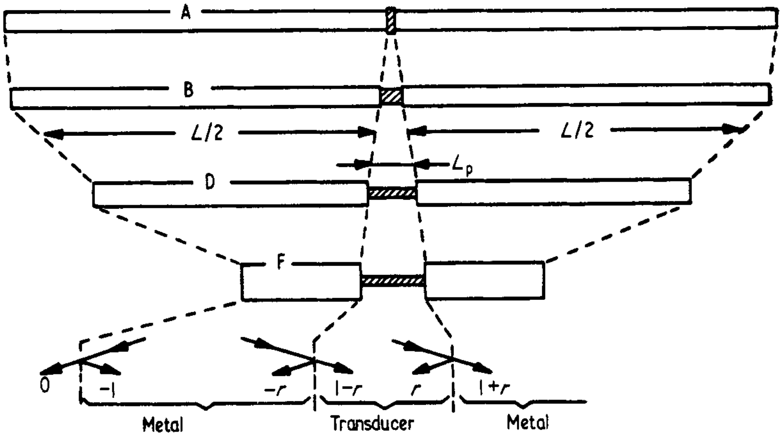

Figure 1. Divided-bar systems having the same fundamental period (T_1), total mass ($M+M_p$), and transducer mass ($M_p = \mu M$). Geometrical aspects are drawn to scale for the material combination aluminium–PXE4, with $\mu = 0{\cdot}05$. Case B is mechanically matched ($r = 0$), whereas A, D, F correspond to $r = -0{\cdot}5$, $+0{\cdot}5$, and $+0{\cdot}8$ respectively. Reflection and transmission coefficients indicated at the bottom refer to force amplitude.

Table 1 displays certain material-independent factors, including the two expressions

$$E_p = \mu\left(\frac{1+r}{1-r}\right)^2\left(1+\frac{\sin\phi}{\phi}\right) \tag{5}$$

$$E = (\cos^2 \tfrac{1}{2}\phi)\{1-(\sin\psi/\psi)\}/\sin^2\tfrac{1}{2}\psi \tag{6}$$

which are proportional to the elastic energies contained in the transducer and in the metal parts.

3. Sensitivity standardization

Gravity-wave excitations will be referred to a hypothetical single pulse of short

† PXE4, a lead–zirconium titanate manufactured by Philips, has elastic properties such that $v_p/v \simeq 0{\cdot}6$, $\rho_p/\rho \simeq 2{\cdot}8$, $Q_p/Q \simeq 10^{-3}$.

Table 1. Characteristics of divided-bar type detectors as a function of transducer section mismatch, r, for different relative transducer masses, μ. Table readings are to be multiplied by units or expressions given in the third column.

Quantity	Equation	Multiplier	$\mu = 0{\cdot}01$						$\mu = 0{\cdot}05$						$\mu = 0{\cdot}2$					
			$r =$ −0·5	−0·1	+0·3	+0·5	+0·7	+0·8	$r =$ −0·5	−0·1	+0·3	+0·5	+0·7	+0·8	$r =$ −0·5	−0·1	+0·3	+0·5	+0·7	+0·8
L_p/L	a	v_p/v	0·003	0·008	0·019	0·030	0·057	0·090	0·017	0·041	0·093	0·150	0·283	0·450	0·067	0·164	0·371	0·600	1·13	1·80
PLF	a	10^{-3}	998	999	1016	1058	1237	1556	989	993	1071	1236	1719	2286	958	975	1189	1471	1972	2354
E_p	(5)	10^{-2}	0·22	1·34	6·9	18·0	64	162	1·11	6·7	34	89	317	798	4·4	26·3	132	341	1210	3048
E	(6)	10^{-3}	999	993	967	925	824	748	994	965	867	776	686	656	969	854	668	601	565	556
Q_{sys}	(11)	Q_p	311	70	14·8	6·1	2·28	1·46	83	15·2	3·5	1·87	1·22	1·08	22·4	4·2	1·50	1·18	1·05	1·02
$\hat{\sigma}_J$	b	(8)	0·58	1·41	3·07	4·63	6·20	5·59	0·58	1·40	2·68	2·97	2·35	1·98	0·59	1·38	1·85	1·57	1·49	1·21
σ_J^{RMS}	b	(8)	0·335	0·82	1·79	2·68	3·66	3·54	0·336	0·81	1·56	1·80	1·58	1·25	0·344	0·79	1·06	0·99	0·81	0·68
σ_{kT}^{RMS}	(10)	(9)	0·335	0·82	1·83	2·87	4·70	5·88	0·341	0·83	1·73	2·38	2·96	3·14	0·363	0·85	1·44	1·64	1·74	1·77
MSNR	(12)	(13)	17·6	8·3	3·77	2·31	1·17	0·73	9·0	3·85	1·69	1·03	0·59	0·42	4·48	1·91	0·90	0·65	0·47	0·39
ESNR	(15)	(16)	0·05	0·11	0·25	0·38	0·51	0·50	0·10	0·25	0·48	0·55	0·48	0·38	0·20	0·45	0·60	0·55	0·45	0·38
n^*	(18)	c	187	36	7·5	3·1	1·1	0·73	43	7·7	1·8	0·93	0·61	0·54	11	2·1	0·75	0·59	0·52	0·51
SNR*	(19)	(20)	0·64	0·69	0·69	0·66	0·55	0·43	0·68	0·69	0·64	0·54	0·38	0·28	0·67	0·66	0·52	0·43	0·33	0·28

a Relative transducer length, $L_p/L = (v_p/v)(\phi/\psi)$, and period length factor, PLF $= \frac{1}{2}T_1/\{(L_p/v_p)+(L/v)\} = \pi/(\phi+\psi)$, using equations (2), (3), (4).
b Peak and RMS values obtained by numerical integrations (see text).
c Table values refer directly to present-day FET amplifiers looking at room-temperature detectors, assuming $(Q_p p_a/\theta)^{1/2}/\pi\kappa \simeq 0{\cdot}5$.

duration with a non-vanishing integral of the Riemann tensor 1010 component

$$J = c^2 \int_{-\delta t}^{+\delta t} R_{1010}(t)\mathrm{d}t \tag{7}$$

(expected magnitude $\sim 10^{-12}\mathrm{s}^{-1}$). The response to a real burst having a zero time integral is related to the J pulse response essentially by a function of (τ/T_1) where τ is the average interval between successive J pulses of opposite sign (Gibbons and Hawking 1971, Maeder 1971a); therefore, T_1 must be kept constant in the sensitivity comparisons.

4. Mechanical response

This is the stress amplitude σ_J developed at the centre of the transducer region during the first half-period following a J pulse, assuming that the detector was initially at rest (as if $kT = 0$). A perfectly matched system ($r = 0$) is similar to a homogeneous cylinder for which the time-domain reponse was discussed by Maeder (1971a); at the centre, the stress waveform is triangular with RMS and peak amplitudes (converted to the transducer material) given by

$$\sigma_J{}^{\mathrm{RMS}}(0) = \frac{\hat{\sigma}_J(0)}{\sqrt{3}} = \frac{v}{v_\mathrm{p}}\frac{JY_\mathrm{p}T_1}{4\sqrt{3}} \tag{8}$$

where the argument (0) refers to the r value. The triangular waveform results from a ramp function which is successively reflected at the cylinder end faces. For $r \neq 0$, multiple reflections were calculated on a computer to obtain detailed waveforms; condensed results, $\hat{\sigma}_J(r)$ and $\sigma_J{}^{\mathrm{RMS}}(r)$, are reported in table 1 using expression (8) as a convenient unit.

5. Thermal response

Thermal response of a perfectly matched system:

$$\sigma_{kT}{}^{\mathrm{RMS}}(0) = (2\theta Y_\mathrm{p}\rho_\mathrm{p}/M_\mathrm{sys})^{1/2} \tag{9}$$

where $\theta \equiv kT$. For $r \neq 0$, (9) has to be multiplied by

$$\sigma_{kT}{}^{\mathrm{RMS}}(r)/\sigma_{kT}{}^{\mathrm{RMS}}(0) = [\{(1+\mu)/(E_\mathrm{p}+E)\}^{1/2}. \tag{10}$$

6. Mechanical signal-to-noise ratio

Mechanical signal-to-noise ratio (MSNR) is based on the theoretical limit of filter gain (assuming zero amplifier noise) which may be increased to $(2Q/\pi)^{1/2}$ using a half-period resolving time (Maeder 1971b). The mechanical Q_sys of the detectors considered in this letter depends essentially on Q_p according to

$$Q_\mathrm{sys}/Q_\mathrm{p} \simeq (E_\mathrm{p}+E)/\{E_\mathrm{p}+(Q_\mathrm{p}/Q)E\} \tag{11}$$

provided that the bonding material (subscript b, thickness L_b) has a negligible effect;

this requires $L_b \ll L_p Q_b / Q_p$. The final expression includes a factor $n^{-1/2}$ (see equation (17)) to allow for a compromise if required by actual amplifier noise:

$$\text{MSNR} = (2Q_{sys}/\pi n)^{1/2} \sigma_J^{RMS}(r)/\sigma_{kT}^{RMS}(r). \tag{12}$$

Typical results ($Q_p/Q = 10^{-3}$, table 1) are given in units of

$$K_m = \left(\frac{2Q_p}{\pi n}\right)^{1/2} \frac{\text{expression (8)}}{\text{expression (9)}} = \tfrac{1}{4} vJT_1 \left(\frac{M_{sys} Q_p}{3\pi n \theta}\right)^{1/2}. \tag{13}$$

7. Electric signal-to-noise ratio

Electric signal-to-noise ratio (ESNR) denotes a sensibility limit that could theoretically be attained with a perfectly noise-free detector ($\theta = 0$). The electronic detection limit, known in terms of energy (Maeder 1972), is to be compared with the time-averaged total elastic energy in the transducer, given by

$$W_J(r) = \frac{1}{2Y_p} (\sigma_J^{RMS}(r))^2 \{1 + (\sin\phi/\phi)\} M_p/\rho_p. \tag{14}$$

With respect to amplitudes, ESNR $\propto W_J^{1/2}$ so that

$$\text{ESNR} = (\sigma_J^{RMS}(r)/\sigma_J^{RMS}(0))[\mu\{1 + (\sin\phi/\phi)\}/(1+\mu)]^{1/2} K_e \tag{15}$$

with a proportionality constant K_e set by electrical matching considerations; it can be shown that

$$K_e \simeq \tfrac{1}{4} \kappa vJT_1 (\pi n M_{sys}/3p_a)^{1/2} \tag{16}$$

where κ = piëzoelectric coupling coefficient ($\simeq 0{\cdot}7$), p_a = amplifier noise energy, and

$$n = 1, 2, 3, \ldots = \text{resolving time in units of } \tfrac{1}{2}T_1 \tag{17}$$

depending on the choice of filter parameters (Maeder 1971b).

8. Optimum geometry

To maximize the overall signal-to-noise ratio (SNR), n is chosen equal to the integer nearest to

$$n^* = \frac{\text{MSNR table value}}{\text{ESNR table value}} \times \frac{(Q_p p_a/\theta)^{1/2}}{\pi\kappa} \tag{18}$$

or 1, whichever is greater. $n = n^*$ would provide MSNR $\equiv$ ESNR, resulting in a theoretical limit,

$$\text{SNR}^* = \text{Max}\left(\frac{\text{MSNR} \times \text{ESNR}}{(\text{MSNR}^2 + \text{ESNR}^2)^{1/2}}\right) = \left(\frac{\text{MSNR} \times \text{ESNR}}{2}\right)^{1/2}. \tag{19}$$

Numerical values (table 1) given in units of

$$K^* = (K_m K_e)^{1/2} = \tfrac{1}{4} vJT_1 (\kappa M_{sys}/3)^{1/2} (Q_p/p_a\theta)^{1/4} \tag{20}$$

go through flat maxima, for example at $r_{opt} = -0{\cdot}2$ for $\mu = 0{\cdot}05$; however, $r = 0$ always yields at least 95% of the maximum SNR* for a given μ.

9. Conclusions

Any design with mechanically matched transducers is satisfactory, although there is no advantage in using transducer masses greater than 1% of $M_{\rm sys}$. Tabulated n^* values indicate an optimum resolving time of about five periods, using present day FET amplifiers with detectors at room temperature. If better amplifiers become available, time resolution can be improved as well as the SNR*. If detectors are cooled without improving the amplifiers, time resolution must be compromised in order to achieve a sensibility improvement. Neglecting the MSNR–ESNR-optimization, previous authors generally arrived at sensibility formulae proportional to $K_{\rm m}$ (equation (13)), or $\theta^{-1/2}$. The new result (equation (20)) shows a temperature dependence proportional to $\theta^{-1/4}$, with the understanding that any variations of amplifier and transducer characteristics should be evaluated separately.

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References

Aplin P S 1972 *Contemp. Phys.* **13** 283–93
Douglass D H and Tyson J A 1971 *Nature, Lond.* **229** 34–6
Gibbons G W and Hawking S W 1971 *Phys. Rev.* D **4** 2191–7
Maeder D G 1971a *Z. angew. Math. Phys.* **22** 1146–54
——1971b *Electron. Lett.* **7** 767–9
——1972 *Electron. Lett.* **8** 198–9
Rasband S N, Pipes P B, Hamilton W O and Boughn S P 1972 *Phys. Rev. Lett.* **28** 253–5
Weber J 1970 *Lett. Nuovo Cim.* **4** 653–8